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ON CENTRAL 2-SYLOW INTERSECTIONS[†]

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ABSTRACT
In this paper we classify finite groups, in which a center of a Sylow 2-subgroup is contained in no more than six distinct Sylow 2-subgroups.

1. Introduction

It is well known that if the center Z, of a Sylow 2-subgroup S, of the finite group G, is contained in no more Sylow 2-subgroups of G, then being weakly closed in S, Z is strongly closed in S with respect to G by corollary 5.3 of [1].

THEOREM. *Let G be a finite group in which the center of S, a Sylow 2-subgroup of G, is contained in no more than six distinct Sylow 2-subgroups of G.*

Then one of the following holds: either

(i) *there exists a non-trivial abelian subgroup of S, strongly closed in S with respect to G, or*

(ii) *G has a subgroup of index 2, or*

(iii) *S is isomorphic to a dihedral group of order 8, or to a quasi-dihedral group of order 16.*

If the center of S is contained in no more than four distinct Sylow 2-subgroups of G, then (i) *holds.*

We obtain as corollaries two characterization theorems for finite simple groups.

COROLLARY 1. *Let G be a finite non-abelian simple group in which a center of a Sylow 2-subgroup of G is contained in no more than six distict Sylow 2-subgroups of G.*

Then G is isomorphic to one of the following groups: $L_2(q), U_3(q), S_z(q)$, q even,

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$L_2(q)$, $q \equiv 3, 5 (\mathrm{mod}\, 8)$, $L_2(7)$, $L_2(9)$, $L_3(3)$, M_{11}, J_1, or a simple group of Ree type of characteristic 3.

COROLLARY 2. *Let G be a finite non-abelian simple group in which a center of a Sylow 2-subgroup of G is contained in no more than four distinct Sylow 2-subgroups of G.*

Then G is isomorphic to one of the following groups: $L_2(q)$, $U_3(q)$, $S_z(q)$, q even, $L_2(q)$, $q \equiv 3, 5 (\mathrm{mod}\, 8)$, J_1, or a simple group of Ree type of characteristic 3.

These results and many aspects of the proof are generalizations of those in [6].

2. Notations, definitions and preliminary results

Most of the notations are standard and might be found in [5]. Other notations and definitions are mainly those of [6], with minor modifications. We repeat them here for the convenience of the reader.

Thus, all groups considered are finite, $H \leqq G (H < G)$ means that H is a (proper) subgroup of G, $\mathrm{Syl}_p(G)$ denotes the set of all Sylow p-subgroups of G. For X, a subgroup of G, denote by $O'(X)$ the subgroup of X, generated by the elements of odd order. By $A \equiv B$ we mean that A is defined by B.

For P, a p-subgroup of G, we define $\Sigma(P) \equiv \{S \in \mathrm{Syl}_p(G) \mid P \leqq S\}$, $\sigma(P) \equiv |\Sigma(P)|$ and $\bar{P} = \cap \Sigma(P)$. We denote by ζ the conjugacy class of centers of Sylow 2-subgroups of G, and for $Z \in \zeta$ we denote $\bar{Z}$ by $T(Z)$, and set $\tau \equiv \{T(Z) | Z \in \zeta\}$.

For any subgroup X, of G, define $\zeta_X \equiv \{Z \in \zeta \mid Z \leqq X\}$, and $\tau_X \equiv \{T \in \tau \mid T \leqq X\}$. Define $\tau' = \{T \in \tau \mid$ for every $S, S' \in \Sigma(T)$, $S \neq S'$ implies that $S \cap S' = T\}$. Finally, let D_n, QD_n and E_n denote the dihedral, quasi-dihedral and elementary abelian groups of order n, respectively.

Some immediate consequences of these definitions are summarized in the following lemmas.

LEMMA 1. (i) *If P, P' are p-subgroups of G such that $P = \bar{P}$ and $P < P'$ then $N_{P'}(P)/P$ acts faithfully on $\Sigma(P)$. If, moreover, $P < X \leqq P'$ implies $\bar{X} = P'$, then $N_{P'}(P)/P$ acts faithfully and fixed point freely on $\Sigma(P) \setminus \Sigma(P')$.*

(ii) *Let P be a p-subgroup of G. Then $\sigma(P) \equiv 1 (\mathrm{mod}\, p)$.*

PROOF. (i) is a restatement of [6] lemma 4, and (ii) is just [6] lemma 6.

LEMMA 2. (i) *The set τ is a conjugacy class of 2-subgroups of G.*

(ii) *If $T \in \tau$, $Z \in \zeta$ and $Z \leqq T$, then $T = T(Z)$.*

(iii) *If* $T \in \tau$ *and* $S \in \mathrm{Syl}_2(G)$, *then* $S \in \mathrm{Syl}_2(N_G(T))$ *iff* $Z(S) \leqq T$,

(iv) $|N_G(T)|_2 = |G|_2$.

PROOF. (i) Let $T_1, T_2 \in \tau$, say, $T_i \equiv T(Z(S_i))$, $S_i \in Syl_2(G)$, $i = 1,2$. There exists some $g \in G$ such that $S_1^g = S_2$, hence $T_1^g = (\cap \Sigma(Z(S_1)))^g = \cap \Sigma(Z(S_1))^g = \cap \Sigma(Z(S_1^g)) = \cap \Sigma(Z(S_2)) = T_2$.

(ii) For $Z' \in \zeta$, $T' \equiv T(Z')$ we have that $\sigma(T') = \sigma(Z')$. Thus by (i), $Z \in \zeta$, $T \in \tau$ imply that $\sigma(T) = \sigma(Z)$. Hence, if in addition $Z \leqq T$, the fact that $\Sigma(T) \subseteq \Sigma(Z)$ implies that $\Sigma(T) = \Sigma(Z)$, whence $T = \cap \Sigma(T) = \cap \Sigma(Z) = T(Z)$.

(iii) Let $T \in \tau$ and $S \in \mathrm{Syl}_2(G)$. If $Z \equiv Z(S) \leqq T$, then by (ii) $T = T(Z)$, hence if $g \in N_G(Z)$, then $T^g = T(Z)^g = T(Z^g) = T(Z) = T$, so that $S \subseteq N_G(Z) \subseteq N_G(T)$, whence $S \in \mathrm{Syl}_2(N_G(T))$ as required.

Now, as $T \in \tau$, we know that there exists some $S' \in \mathrm{Syl}_2(\mathrm{G})$ such that $T = T(Z(S'))$. By what we have just proved, $S' \in \mathrm{Syl}_2(N_G(T))$, so $|N_G(T)|_2 = |G|_2$ proving (iv).

Moreover, assuming $S \in \mathrm{Syl}_2(N_G(T)) \subseteq \mathrm{Syl}_2(G)$ we have some $g \in N_G(T)$ such that $S = (S')^g$. Thus $Z(S) = Z(S'^g) = Z(S')^g \subseteq T^g = T$, and we are through.

LEMMA 3. *Let* T *be a* 2-*subgroup of* G *satisfying* $T = \bar{T}$ *and* $\sigma(T) \leqq 6$. *Then one of the following holds*:

(i) *For every* $S, S' \in \Sigma(T)$, $S \neq S'$ *implies* $S \cap S' = T$, *or*

(ii) $\sigma(T) = 5$, *and denoting* $\Sigma(T) = \{S_i\}_{i=1}^5$, *we have* (*after renumbering the* S_i's *if necessary*) *that*

(1) $\{N_{S_1}(T)\} = \mathrm{Syl}_2(N_G(T))$, $|N_{S_i}(T) : T| = 2$ *for* $i > 1$,

(2) $T_1 \equiv S_1 \cap S_2 \cap S_3 = S_1 \cap S_2 = S_1 \cap S_3 = S_2 \cap S_3 > T$,
$T_2 \equiv S_1 \cap S_4 \cap S_5 = S_1 \cap S_5 = S_4 \cap S_5 = S_1 \cap S_4 > T$, *and*
$S_i \cap S_j = T$ *for all* $i = 2,3$, $j = 4,5$,

(3) $N_{T_1}(T) = N_{S_2}(T) = N_{S_3}(T)$, $N_{T_2}(T) = N_{S_4}(T) = N_{S_5}(T)$,

(4) *for* $i = 1,2$ $\sigma(T_i) = 3$, $|N_{S_1}(T_i) : T_i| = 2$, $N_G(T_i)$ *is transitive on* $\Sigma(T_i)$, $N_{S_1}(T_1)/T_1$ *interchanges* S_2 *and* S_3 *by conjugation and* $N_{S_1}(T_2)/T_2$ *interchanges* S_4 *and* S_5 *by conjugation*,

(5) $N_S(T)/T \tilde{\subset} D_8$.

Thus if $\sigma(T) \leqq 4$, *only* (i) *can happen.*

PROOF. By Lemma 1(ii), $\sigma(T) = 1, 3$ or 5. If $\sigma(T) = 1$, then (i) holds vacuously. If $\sigma(T) = 3$, then (i) holds by [6] corollary 7. If $\sigma(T) = 5$ and (i) does not hold, we use the proof of [6] lemma 11, which, except for the last paragraph, is valid under our assumptions, to prove (ii) (1)–(3). The last assertions, (4), (5), are easy consequences of (1)–(3) and Lemma 1(i).

Following [1], we now construct the variant of weak conjugation family, suitable for our purposes. Let S be a Sylow p-subgroup of G. We define $\kappa = \{K \leqq S \mid Z(S) \leqq K, K = \bar{K}\}$. Theorem 5.2 of [1] asserts that κ is a weak conjugation family. We restate it as

LEMMA 4. *Let S be a Sylow p-subgroup of a finite group G.*

Define $\kappa = \{K \leqq S \mid Z(S) \leqq K, K = \bar{K}\}$. Then, if $x, y \in S$ and x is conjugate to y in G, there exist $K_i \in \kappa$, $u_i \in N_G(K_i)$, $i = 1, \cdots, n$ such that $x^{u_1 \cdots u_n} = y$. If, moreover, for some $1 \leqq i \leqq n$, $K_i < S$, we may assume that u_i is a p-element.

The following argument is an analogue of that of [3]. Let S, S' be Sylow p-subgroups of G. We denote $S \sim S'$ iff $\zeta_{S \cap S'} \neq \varnothing$, and $S \approx S'$ if there exist $\{S_i\}_{i=1}^n \subseteq \mathrm{Syl}_p(G)$ such that $S = S_1$, $S_n = S'$, and $S_i \sim S_{i+1}$ for every $1 \leqq i \leqq n-1$. It is clear that the relation $\approx$ is an equivalence relation on $\mathrm{Syl}_p(G)$.

LEMMA 5. *In the permutation representation of G acting on $\mathrm{Syl}_p(G)$ by conjugation, the $\approx$-equivalence classes form a system of imprimitivity. Denoting by H the stabilizer of some $\approx$-equivalence class C we have:*

(i) $C = \mathrm{Syl}_p(H)$,

(ii) *If K is a p-subgroup of H such that $\zeta_K \neq \varnothing$, then $N_G(K) \leqq H$,*

(iii) *If H is a p-local subgroup of G, then for every $S \in \mathrm{Syl}_p(H)$, $Z(O_p(H))$ is a non-trivial abelian subgroup strongly closed in S with respect to G.*

PROOF. Except for (iii) the proof is a repetition of [3] lemma 2.1. As for (iii), we claim that if $x, y \in S$, and x is conjugate to y in G, then x is conjugate to y in H, whence $Z(O_p(H))$, which is a non-trivial abelian normal subgroup of H, is a strongly closed abelian subgroup of S with respect to G.

Indeed, let K be a p-subgroup in κ, the weak conjugation family defined above, and let $u \in N_G(K)$. As $Z(S) \leqq K$, we have that $\zeta_K \neq \varnothing$, hence the element u, permuting the subgroups of ζ_K, must stabilize C, whence $u \in H$. Thus by Lemma 4, if $x, y \in H$ and x is conjugate to y in G, x is conjugate to y in H, and we are through.

Let x_0 be a characteristic subgroup of y_0, and let ξ, η, denote their G-conjugacy classes respectively. For every $y \in \eta$, we can define $x(y)$ to be the unique subgroup of ξ, satisfying $x(y) = x_0^g$ for some g such that $y = y_0^g$. Indeed, if for some $g' \in G$, $y = y_0^{g'}$, then $g'g^{-1} \in N_G(y_0) \subseteq N_G(x_0)$, and $x_0^{g'} = x_0^g$. Moreover, $x: \eta \to \xi$ is the unique function from η to ξ such that $x(y_0) = x_0$, and $x(y^g) = (x(y))^g$ for every $g \in G$.

This discussion motivates the following definition. Let x_0, y_0 be subgroups of the group G, and let ξ, η denote their G-conjugacy classes respectively. We say that x_0 is kind of characteristic in y_0 with respect to G, if $N_G(y_0) \subseteq N_G(x_0)$. If such is the case, then, as discussed above, we can define a unique function $x:\eta \to \xi$ such that $x(y_0) = x_0$, and $x(y^g) = (x(y))^g$ for every $y \in \eta$, $g \in G$.

LEMMA 6. *Let S be a Sylow 2-subgroup of X and H, the stabilizer in G of C, its $\approx$-equivalence class.*

If X_0 is kind of characteristic in S with respect to G such that for every $S' \in \Sigma(Z(S))$, $X(S') = X_0$, then X_0 is normal in H.

PROOF. It suffices to prove that if $S_1, S_2 \in C$ satisfy $S_1 \sim S_2$, then $X(S_1) = X(S_2)$, for then, if $S' \in \mathrm{Syl}_2(H) = C$, we have that $S \approx S'$ so that $X(S') = X_0$, whence H, permuting $\mathrm{Syl}_2(H)$, must normalize X_0.

Hence let $S_1, S_2 \in C = \mathrm{Syl}_2(H)$ satisfy $Z(S_3) \subseteq S_1 \cap S_2$ for some $S_3 \in \mathrm{Syl}_2(G)$. As $S_3 \in \mathrm{Syl}_2(H)$, there exists some $g \in H$ such that $S_3 = S^g$, whence $S_1^{g^{-1}}$, $S_2^{g^{-1}} \in \Sigma(Z(S))$, so that $X(S_1^{g^{-1}}) = X_0 = X(S_2^{g^{-1}})$. Hence $X(S_1) = X(S_1^{g^{-1}})^g = X(S_2^{g^{-1}})^g = X(S_2)$, and we are through.

LEMMA 7. *If $\tau = \tau'$, then S, a Sylow 2-subgroup of G, contains a non-trivial abelian subgroup, strongly closed in S with respect to G.* .

PROOF. Let C denote the $\approx$-equivalence class of S, and H the stabilizer of C in G. Clearly $T \equiv T(Z(S))$ is kind of characteristic in S. We claim that $T(Z(S')) = T(Z(S))$ for every $S' \in \Sigma(S)$. Indeed, let $S' \in \Sigma(Z(S))$, and assume, by way of contradiction, that $Z(S') \not\subseteq T$. Thus $T < \langle T, Z(S')\rangle \leqq N_{S'}(T) \leqq S''$ for some $S'' \in \mathrm{Syl}_2(N_G(T)) \subseteq \mathrm{Syl}_2(G)$. Hence $S', S'' \in \Sigma(T)$ and $T < S' \cap S''$ imply, as $\tau = \tau'$, that $S' = S'' \in \mathrm{Syl}_2(N_G(T))$, whence by Lemma 2 (iii) we must have $Z(S') \subseteq T$. Therefore our claim is proved, and we may turn to Lemma 6, and then to Lemma 5(iii) to arrive at the desired conclusion.

REMARK 8. The corollaries are easy consequences of the theorem once we quote the main theorem of [4]. Then, all we have to do is to prove that if S, a Sylow 2-subgroup of G, is isomorphic to a dihedral group of order 8 or to a quasi-dihedral group of order 16, then G, a finite simple group satisfying the assumption of the theorem, must be isomorphic to either one of $L_2(7)$, $L_3(3)$ or M_{11}.

This can be done either directly, or, remembering that $|Z(S)| = 2$, and that G must have just one conjugacy class of involutions (by (7.7.3) of [5], and (7) ex. 7 of [5]), we can quote theorem 3 of [6].

3. Proof of the theorem

Let G be a finite group satisfying the hypothesis of the theorem. We prove that G satisfies the conclusion of the theorem in a series of lemmas. Let S be a Sylow 2-subgroup of G, $Z \equiv Z(S)$ and $T \equiv T(Z)$. Let H be the stabilizer in G of C, the $\approx$-equivalence class of S. In the following lemma we gather several properties needed for the proof of the theorem, which G may be assumed to have.

LEMMA 9. *We may assume that*:

(i) $O_2(H) = 1$,

(ii) $\tau \neq \tau'$,

(iii) $\sigma(T) = 5$, $\Sigma(T) = \{S_i\}_{i=1}^5$, $\{S_1\} = \mathrm{Syl}_2(N_G(T))$,

(iv) $T_1 \equiv S_1 \cap S_2 \cap S_3 = S_1 \cap S_2 = S_1 \cap S_3 = S_2 \cap S_3 > T$,
$T_2 \equiv S_1 \cap S_4 \cap S_5 = S_1 \cap S_4 = S_1 \cap S_5 = S_4 \cap S_5 > T$,
$S_i \cap S_j = T$ *for all* $i = 2, 3$, $j = 4, 5$,

(v) $T_1 = N_{S_2}(T) = N_{S_3}(T)$, $T_2 = N_{S_4}(T) = N_{S_5}(T)$, $|T_i : T| = 2$ *for* $i = 1, 2$,
$N_{S_1}(T)/T \tilde{\subset} D_8$

(vi) *for* $i = 1, 2$, $\sigma(T_i) = 3$, $|N_{S_1}(T_i) : T_i| = 2$, $N_G(T_i)$ *is transitive on* $\Sigma(T_i)$,
$N_{S_1}(T_1)$ *interchanges* S_2 *and* S_3, $N_{S_1}(T_2)$ *interchanges* S_4 *and* S_5,

(vii) $S_1 = S$, $N_G(S) = N_G(Z) = N_G(T)$,

(viii) $Z_i \equiv Z(S_i)$ *centralizes* T *for every* $1 \leqq i \leqq 5$,

(ix) $\zeta_S = \{Z_i\}_{i=1}^5$, $\zeta_T = \{Z\}$, $\zeta_{T_1} = \{Z_1, Z_2, Z_3\}$, $\zeta_{T_2} = \{Z_1, Z_4, Z_5\}$,

(x) $R_1 \equiv Z_1 \cap Z_2 \cap Z_3 = Z_1 \cap Z_2 = Z_1 \cap Z_3 = Z_2 \cap Z_3$,
$R_2 \equiv Z_1 \cap Z_4 \cap Z_5 = Z_1 \cap Z_4 = Z_1 \cap Z_5 = Z_4 \cap Z_5$,
$U_1 \equiv T(Z_1) \cap T(Z_2) \cap T(Z_3) = T(Z_1) \cap T(Z_2) = T(Z_1) \cap T(Z_3)$
$= T(Z_2) \cap T(Z_3)$,
$U_2 \equiv T(Z_1) \cap T(Z_4) \cap T(Z_5) = T(Z_1) \cap T(Z_4) = T(Z_1) \cap (Z_5)$
$= T(Z_4) \cap T(Z_5)$,

(xi) *for* $i = 1, 2$ U_i *is a normal subgroup of index* 4 *in* T_i.

PROOF. (i) clear by Lemma 5(iii), and (ii) by Lemma 7.

(iii)–(vi) clear by (ii), Lemma 3 and Lemma 2 (iv).

(vii) clearly $N_G(S) \subseteq N_G(Z) \subseteq N_G(T)$. By (iii), $S_1 = S$, and moreover $N_G(T) \subseteq N_G(S)$, whence we are through.

(viii) clear by (iii).

(ix) clear by (iii), (v) and Lemma 2 (iii).

(x) clear by (ix) and (vi).

(xi) clear by (v), (vi) and (x).

LEMMA 10. (i) *The subgroup* $X \equiv O'(N_G(Z))$ *is kind of characteristic in* S *with respect to* G,

(ii) *for every* $S' \in \Sigma(Z)$, $O'(N_G(Z(S')) = X$,

(iii) $N_G(S) = S \times X$,

(iv) *if* $x, y \in Z$, *then* x *is conjugate to* y *in* G *only if* $x = y$.

PROOF. (ii) Let x be an element of $N_G(Z) = N_G(S)$ of odd order. As x normalizes S, x acts on $\{T_1, T_2\}$, and being of odd order, it acts trivially, normalizing both T_1 and T_2. Now, x acts on $\{Z_i\}_{i=2}^{5}$, and so, normalizing T_1 and T_2, x acts on $\{Z_2, Z_3\}$, and on $\{Z_4, Z_5\}$. Thus being of odd order, x normalizes Z_i for all $1 \leqq i \leqq 5$. Hence for every $S' \in \Sigma(Z)$, $X \subseteq O'(N_G(Z(S')))$, so clearly $O'(N_G(Z(S'))) = X$ as required.

As (i) is obvious, we know by Lemma 6 that X is normal in H, whence by Lemma 9(i), $X \cap S = O_2(X) = 1$. Thus $N_G(S) = N_G(Z) = S \times X$, and (iii) is proved.

Now, (iv) is immediate by (iii) and lemma 4.5 of [2].

LEMMA 11. (i) *If* $x, y \in T$, *and* x *is conjugate to* y *in* G, *then* x *is conjugate to* y *in* S,

(ii) *for* $1 \leqslant i \leqslant 5$, $Z_i \cap T = Z_i \cap Z$,

(iii) $Z \cap U_1 \cap U_2 = R_1 \cap R_2 = \cap \zeta_S$.

PROOF. (i) It is clear that $\langle \zeta_S \rangle$ is weakly closed in S with respect to G. Hence, as $T \subseteq C_S(\langle \zeta_S \rangle)$ by Lemma 9 (viii), (i) is immediate by lemma 4.5 of [2] and Lemma 10 (iii).

(ii) Let $x \in Z_i \cap T$. As Z_i is conjugate to Z_1 in G, there exists some $y \in Z_1$ conjugate to x in G. Hence, by (i), x is conjugate to y in S, whence $x \in Z_i \cap Z_1$. Thus (ii) is proved.

(iii) By (ii) and Lemma 9 (vi) $Z \cap T(Z_i) = Z \cap Z_i$ for $1 \leqq i \leqq 5$, whence $Z \cap U_1 \cap U_2 = R_1 \cap R_2 = \cap \zeta_S$.

LEMMA 12. *The conclusion of the theorem holds.*

PROOF. The subgroup $\cap \zeta_S$ is centralized by $N_G(S) = N_G(T)$, and by every 2-element of $N_G(T_i)$, $i = 1, 2$. Hence by Lemma 4, $\cap \zeta_S$ is strongly closed in S with respect to G. Thus being abelian, we may assume that $\cap \zeta_S = 1$.

Now, $U_1 \cap U_2$ is normal in S, so $Z \cap U_1 \cap U_2 = \cap \zeta_S = 1$ forces $U_1 \cap U_2 = 1$. As we have for $i = 1, 2$ $|T : U_i| \leqq 2$, there are two possibilities only:

(i) $U_1 = U_2 = 1$.

Then, $T = Z \simeq E_2$, and setting $Z_i = \langle z_i \rangle$ for $1 \leqq i \leqq 5$, we have that $C_S(z_2) =$

$\langle z_1, z_2\rangle \simeq E_4$ whence S is dihedral or quasi-dihedral by lemma 4 of [7]. As $S/T \tilde{\subset} D_8$ and $T \simeq E_2$, we must have $S \simeq D_8$, QD_{16}, or D_{16}.

Now as $|Z(S)| = 2$, every involution in S, conjugate to z_1, must be in $\{z_i\}_{i=1}^5$ as $\zeta_S = \{Z_i\}_{i=1}^5$. Thus if $S \simeq D_{16}$, S has more than one conjugacy class of involutions. As z_1, the central involution of S, is conjugate to one of $\{z_i\}_{i=2}^5$, S has exactly two conjugate classes of involutions, and we are through by (7.7.3) (ii) of [5].

(ii) $U_i \neq 1$ for some $i = 1, 2$, say, $U_1 \neq 1$.

In this case, as $|T : U_2| \leqq 2$ for $i = 1, 2$, $U_i \tilde{\subset} E_2$ forces $T = U_1 \times U_2 \tilde{\subset} E_4$.

Assume first that T_1 is normal in S. In that case T_2 is also normal in S, and $S/T \simeq E_4$ by Lemma 9(v), (vi), say, $S/T = \{T, T_1/T, T_2/T, T_3/T\}$. Also, for $i = 1, 2$, U_i is normal in S, so, as $U_1 \neq 1$, $R_1 = U_1 \cap Z \neq 1$. Hence $T = Z \simeq E_4$, $T_1 \simeq T_2 \simeq E_8$ and $R_1 \simeq R_2 \simeq E_2$.

If there exists an involution in $T_3 - T$, then $T_3 \simeq E_8$, so as $S \bigcup_{i=1}^3 T_i$, S consists of the identity and involutions only. Thus $S \simeq E_{16}$, in contradiction to the fact that $Z \simeq E_4$. Thus every involution of S lies in $\bigcup_{i=1}^2 T_i = \bigcup_{i=1}^5 Z_i$.

Let us analyse the involution fusion pattern in S, and first let us fix some notation. Let $\{1, t, z, zt\} \equiv Z$, $\langle t\rangle \equiv R_1$, and $\langle zt\rangle \equiv R_2$. Denote by C_1, C_2, C_3 the intersection of S with the G-conjugacy classes of t, z and zt, respectively. Clearly, every involution of S lies in $\bigcup_{i=1}^3 C_i$. Moreover, by Lemma 10(iv), $|C_i \cap Z_j| = 1$ for every $1 \leqq i \leqq 3$, $1 \leqq j \leqq 5$. Now denote $\{u\} \equiv C_2 \cap Z_2$, $\{v\} \equiv C_1 \cap Z_4$ to get: $Z = \{1, t, z, zt\}$, $Z_2 = \{1, t, u, ut\}$, $Z_3 = \{1, t, uz, uzt\}$, $Z_4 = \{1, v, vzt, zt\}$, $Z_5 = \{1, vz, vt, zt\}$, where every non-identity element is an involution, and these are all the involutions of S.

Clearly, $ut \in C_3$ and $vzt \in C_2$. If $vt \in C_1$, then clearly $C_1 = \{t, v, vt\}$, whence $\{1, t, v, vt\}$ is an abelian subgroup, strongly closed in S with respect to G. Hence we may assume that $vt \in C_2$, whence $vz \in C_1$. Analogously, if $uz \in C_3$, then $C_3 = \{zt, ut, uz\}$, whence $\{1, zt, ut, uz\}$ is an abelian subgroup, strongly closed in S with respect to G, and we may assume $uz \in C_2$, so $uzt \in C_3$. Thus $C_1 = \{t, v, vz\}$, $C_2 = \{z, u, uz, vzt, vt\}$, and $C_3 = \{zt, ut, uzt\}$.

Concluding, as v interchanges Z_2 and Z_3 by conjugation, $u \in C_1 \cap Z_2$ implies $u^v \in C_1 \cap Z_3 = \{uz\}$. Thus $(uv)^2 = z$, whence $\langle u, v\rangle = \{1, u, v, uv, z, uz, vz, uvz\} \simeq D_8$. As $|S : \langle u, v\rangle| = 2$, and $\langle u, v\rangle \cap C_3 = \varnothing$, lemma 5.38 of [8] tells us that G has a subgroup of index 2.

To conclude, assume that T_1 is conjugate to T_2 in S. In that case $U_1 \simeq U_2 \simeq Z_2$, $T \simeq E_4$, $Z \simeq E_2$. Now, $T_1 \subseteq C_S(T)$, and as $T > Z$, $C \equiv C_S(T) = T_1T_2$ is of index 2 in S. Hence every $v \in S\backslash Z$, acts the same on T, that is: v centralizes Z, and if $t \in T\backslash Z$, then $(tt^v)^v = tt^v$(as $v^2 \in C$) implies ($v \notin C$) that $tt^v = z$, so that $t^v = tz$.

We claim that there exists some involution $v \in S\backslash C$. Let vT be an involution of S/T not in C/T. As $S/T \simeq D_8$ we must have $v \not\in C$, but $v^2 \in T$. Assuming that v is not an involution, we have that $v^2 \in T$ centralized by v must equal z. Hence taking $v' \equiv vt$ for some $t \in T \backslash Z$ we have that $v' \in S \backslash C$, and that $(vt)^2 = vtvt = vtv^{-1}v^2t = t^v zt = tzzt = 1$, so that v' is the required element, in which case we rename it v.

We claim that v cannot be conjugate in G to any involution in C. Indeed, according to Lemma 4 this can be achieved in a finite series of steps, each of which is a conjugation in the normalizer of some $K \in \kappa$. Now $\kappa = \{T, T_1, T_2, S\}$, and S is the only subgroup in κ to which v belongs. But every $g \in N_S(T)$ normalizes $C \equiv C_S(T)$; so v cannot be conjugate in G to any involution in C.

By lemma 5.38 of [8], G has a subgroup of index two, and the theorem is proved.

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